Hypercontractions on Banach space (Theory of operator means and related topics)

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Hypercontractions on Banach space

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1 Introduction

The operator T on a Hilbert space H is an n-hypercontraction for some positive integer n as in Agler [2], if for all $1 \leq m \leq n$,

$$\beta_m(T) := \sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^k \geq 0$$

or equivalently, for all $1 \leq m \leq n$,

$$\langle \beta_m(T)h, h\rangle = \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left\|T^k h\right\|^2 \geq 0 \text{ for all } h \in H.$$

Inspired by the above definition of n-hypercontractions and the work of m-isometries on Hilbert spaces [3] [4] and recent work on (m,p)-isometries on a Banach space X [8] [6] [14] [13], we introduce (m,p)-hypercontractions on X. Let $p \in [1,\infty)$ and let $B(X)$ be the algebra of all bounded linear operators on X. An operator $T \in B(X)$ is called an (m,p)-contraction if

$$\beta_{(m,p)}(T,x) := \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left\|T^k x\right\|^p \geq 0 \text{ for all } x \in X. \tag{1}$$

We say T is an (n,p)-hypercontraction if T is an (m,p)-contraction for all $1 \leq m \leq n$. An operator T is an (m,p)-isometry if $\beta_{(m,p)}(T,x) = 0$ for all $x \in X$. We note that an (m,p)-isometry is automatically an $(m+1,p)$-isometry, see

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formula (4) below. But an (m,p)-hypercontraction is in general not an $(m+1,p)$-hypercontraction. When $n=1$, the power p is irrelevant and a $(1,p)$-contraction is just a contraction. When $n>1$, the power p is highly relevant. For example, it was proved in [8] that there is no $(2,2)$-isometric weighted shifts on l_p for $p \neq 2$. A characterization of (m,q)-isometric weighted shifts on l_p spaces is given by one of the authors in [13]. One of the results in [13] states that if a weighted shift on l_p is an (m,q)-isometry, then $q=pk$ for some integer k.

The following result is well-known [10] [16] [9].

THEOREM A. *Let S denote the unilateral (unweighted) shift of multiplicity one and let $S^{*(\infty)}$ be the backward shift of infinite multiplicity. Let $T \in B(H)$. Then T is unitarily equivalent to a part of $S^{*(\infty)}$ if and only if $\|T\| \leq 1$ and $T^k \to 0$ strongly.*

Agler in [1] developed a C^*-algbera method for operator models and proved an analog of Theorem A with S replaced by Bergman shift B.

THEOREM B. *Let $T \in B(H)$. Then T is unitarily equivalent to a part of $B^{*(\infty)}$ if and only if $I - 2T^*T + T^{*2}T \geq 0$ and $T^k \to 0$ strongly.*

To state the more general result in Agler [2], we need to introduce some notations. Let n be a fixed positive integer.

$$M_n = \left\{ f(z) = \sum_{i=0}^{\infty} \hat{f}(i) z^i : \|f(z)\|_n^2 = \sum_{i=0}^{\infty} (w_{n,i})^{-1} \left|\hat{f}(i)\right|^2 < \infty \right\},$$

where $w_{n,i}$ is defined by

$$w_{n,i} = \binom{n-1+i}{n-1} \text{ so that } (1-z)^{-n} = \sum_{i=0}^{\infty} w_{n,i} z^i, |z|<1. \tag{2}$$

M_n is the Hilbert space of analytic functions on the open unit disc D with the reproducing kernel $k_w(z) = (1-\overline{w}z)^{-n}$. Let S_n be the operator on M_n defined by

$$S_n(f)(z) = zf(z), f \in M_n.$$

Thus S_1 is the unilateral shift on the Hardy space, S_2 is the Bergman shift on the Bergman space and S_n is a weighted shift.

THEOREM C. *Let $T \in B(H)$. Then T is unitarily equivalent to a part of $S_n^{*(\infty)}$ if and only if $\beta_n(T) \geq 0$ and $T^k \to 0$ strongly.*

In this paper, we extend Theorem C to Banach spaces. Recall $l_p(X)$ denote the Banach space defined by

$$l_p(X) = \left\{ f = \{x_i\}_{i=0}^{\infty} : \|f\|^p = \sum_{i=0}^{\infty} \|x_i\|^p < \infty, x_i \in X \text{ for } i \geq 0 \right\}.$$

More generally, we define weighted Banach space $l_{(n,p)}(X)$ by using weight sequences $\{w_{n,i}\}_{i=0}^{\infty}$ as in (2),

$$l_{(n,p)}(X) = \left\{ f = \{x_i\}_{i=0}^{\infty} : \|f\|_n^p = \sum_{i=0}^{\infty} w_{n,i} \|x_i\|^p < \infty, x_i \in X \text{ for } i \geq 0 \right\}.$$

Note $l_p(X) = l_{(1,p)}(X)$. Let B_n be the (unweighted) backward shift on $l_{(n,p)}(X)$ defined by

$$B_n(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots), \{x_i\}_{i=0}^{\infty} \in l_{(n,p)}(X).$$

It is clear that B_1 can be extended to be an invertible bilateral shift defined on two sided $l_p(X)$ space. It is not clear how to extend B_n for $n > 1$. Let $T \in B(X)$. We say T is unitarily equivalent to a part of B_n if there is an isometry W_n from X into $l_{(n,p)}(X)$ such that

$$W_n T = B_n W_n. \tag{3}$$

Note that B_n is invariant on the range $W_n(X)$ and hence one may write

$$T = W_n^{-1} B_n W_n.$$

Now we state the main theorem of this paper.

Theorem 1 *Let $T \in B(X)$. Then T is unitarily equivalent to a part of B_n if and only if T is an (n,p)-contraction and $T^k \to 0$ strongly.*

Instead of working with weighted Banach space $l_{(n,p)}(X)$, we could just work on $l_p(X)$. The trade-off would be that we use weighted backward shift D_n on $l_p(X)$ instead of unweighted backward shift B_n on $l_{(n,p)}(X)$. The operator D_n on $l_p(X)$ is defined by

$$D_n(x_0, x_1, x_2, \cdots, x_i, \cdots) = (c_1 x_1, c_2 x_2, \cdots, c_i x_i, \cdots)$$

where $c_i = (w_{n,i-1}/w_{n,i})^{1/p}, i \geq 1$. The operator D_n is a contraction since $c_i \leq 1$ for all $i \geq 1$. Then Theorem 1 can be reformulated as the following: There is an isometry W_n from X into $l_p(X)$ such that $W_n T = D_n W_n$ if and only if T is an (n,p)-contraction and $T^k \to 0$ strongly.

2 Proof of Theorem 1

The proof of Theorem 1 needs several lemmas which we stated below. Here we will only give the proof of "if" part of Theorem 1 which is short.

We first state a lemma proved on page 2143 in [6].

Lemma 2 *Let $T \in B(X)$, $N \geq n \geq 1$ and $x \in X$. Then*

$$\beta_{(n,p)}(T,x) = \beta_{(n-1,p)}(T,x) - \beta_{(n-1,p)}(T,Tx). \tag{4}$$

We also need the following lemma.

Lemma 3 *Let $T \in B(X)$. If T is an (n,p)-contraction and $T^k \to 0$ strongly, then T is an (n,p)-hypercontraction. Furthermore, for each $x \in X$ and all $0 \leq m \leq n$,*

$$k^m \beta_{(m,p)}(T, T^k x) \to 0 \text{ as } k \to \infty. \tag{5}$$

Lemma 4 *Let $T \in B(X)$, $N \geq n \geq 1$ and $x \in X$. Then*

$$\sum_{k=0}^{N} w_{n,k} \beta_{(n,p)}(T, T^k x) + \sum_{l=0}^{n-1} w_{l+1,N} \beta_{(l,p)}(T, T^{N+1}x) = \|x\|^p \tag{6}$$

The proof of "if" part of Theorem 1. Let $T \in B(X)$ be such that $\beta_{(n,p)}(T,x) \geq 0$ for all $x \in X$ and $T^k \to 0$ strongly. We define W_n from X into $l_{(n,p)}(X)$ as

$$W_n x = \left\{ \beta_{(n,p)}^{1/p}(T, T^i x) \frac{T^i x}{\|T^i x\|} \right\}_{i=0}^{\infty}$$

with the understanding that if $T^i x = 0$ for a specific i, then $\beta_{(n,p)}^{1/p}(T, T^i x) \frac{T^i x}{\|T^i x\|} = 0$. We now show W_n is well-defined and is an isometry. We need to show $\|W_n x\|_n^p = \sum_{i=0}^{\infty} w_{n,i} \beta_{(n,p)}(T, T^i x)$ converges to $\|x\|^p$. By Lemma 4, for $N > n$,

$$\sum_{i=0}^{N} w_{n,i} \beta_{(n,p)}(T, T^i x) = \|x\|^p - \sum_{l=0}^{n-1} w_{l+1,N} \beta_{(l,p)}(T, T^{N+1}x) \leq \|x\|^p.$$

Furthermore for each $0 \leq l \leq n-1$, by Lemma 3 and $\frac{w_{l+1,N}}{(N+1)^l} \to 1$, we have

$$w_{l+1,N} \beta_{(l,p)}(T, T^{N+1}x) = \frac{w_{l+1,N}}{(N+1)^l} (N+1)^l \beta_{(l,p)}(T, T^{N+1}x) \to 0$$

as $N \to \infty$. The proof is complete.

3 A similarity model on Banach space

Theorem 1 gives a characterization of an operator unitarily equivalent to a part of the (n,p)-hypercontraction B_n. What is a characterization of an operator similar to a part of B_n? This question has not even been discussed on Hilbert spaces for $n > 1$. For $n = 1$, the following model theorem of Rota [17] predates Theorem A and is the first example of a universal operator. Let $r(T)$ denote the spectral radius of a bounded operator T.

THEOREM D. *Let* $T \in B(H)$. If $r(T) < 1$, then T is similar to a part of $S^{*(\infty)}$.

The proof of Theorem D and some of its late generalizations (see the book [15]) can be adapted to Banach spaces. Thus some of the results below might be known to experts. Let $T \in B(X)$, we say T is similar to a part of B_n, the backward shift on $l_{(n,p)}(X)$, if there is an bounded operator W_n from X into $l_{(n,p)}(X)$ such that W_n is bounded below and $W_nT = B_nW_n$. The following result is inspired by Proposition 6.6 from [15] and the proof is also similar. However, Proposition 6.6 from [15] only deals with the case $n = 1$.

Theorem 5 *Let $T \in B(X)$. The following statements are equivalent.*
(a) There exist constants $\beta \geq \alpha > 0$ and $Q \in B(X)$, such that for all $x \in X$,

$$\alpha \|x\|^p \leq \sum_{k=0}^{\infty} w_{n,k} \left\|QT^k x\right\|^p \leq \beta \|x\|^p . \tag{7}$$

(b) T is similar to a part of B_n on $l_{(n,p)}(X)$.

Proof. The proof is adapted from the proof of Proposition 6.6 in [15]. Assume (a) holds. We define W_n from X into $l_{(n,p)}(X)$ by

$$W_n x = \left\{QT^k x\right\}_{k=0}^{\infty} .$$

Then assumption (7) is the same as $\alpha \|x\|^p \leq \|Wx\|_n^p \leq \beta \|x\|^p$. Therefore the range of W_n, denoted by $R(W_n)$, is a closed subspace of $l_{(n,p)}(X)$ and W_n from X onto $R(W_n)$ is invertible. It is also clear that $W_nT = B_nW_n$, so $R(W_n)$ is invariant for B_n and

$$T = W_n^{-1}(B_n|R(W_n))W_n.$$

That is, T is similar to the restriction of B_n to $R(W_n)$.

Now assume (b) holds. Let W_n from X into $l_{(n,p)}(X)$ be such that W_n is bounded below and $W_nT = B_nW_n$. Let P_k be the projection from $l_{(n,p)}(X)$ onto its k-th component, $P_k\{x_i\}_{i=0}^{\infty} = x_ke_k$. Let $Q_k = P_kW_n, k \geq 0$. Then for $x \in X$,

$$W_nx = \{Q_kx\}_{k=0}^{\infty}.$$

The relation $W_nT = B_nW_n$ means $Q_kTx = Q_{k+1}x$. Thus $Q_{k+1} = Q_kT$. Set $Q = Q_0$, we have

$$Wx = \{Q_kx\}_{k=0}^{\infty} = \left\{QT^kx\right\}_{k=0}^{\infty}.$$

Now W_n from X into $l_{(n,p)}(X)$ is bounded and bounded below is the same as (7). The proof is complete. ■

The following is the analogue of Theorem D on Banach spaces.

Corollary 6 *Let $T \in B(X)$. If $r(T) < 1$, then T is similar to a part of B_1. Furthermore T is similar to a strict contraction.*

Proof. If $r(T) < 1$, the condition (7) (with $w_{1k} = 1$) holds for by taking Q to be the identity operator. Thus T is similar to a part of B_1. To prove T is similar to a strict contraction, we use the scaling as in [17]. Let ε be such that $r(T) < \varepsilon < 1$. By what we just proved, T/ε is similar to a part of B_1. Therefore T is similar to a part of εB_1. We will show below that $B_1|R(W_1)$ is in fact a strict contraction. ■

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References

[1] J. Agler, The Arveson extension theorem and coanlytic models, Integral Equations Operator Theory 5(1982) 608-631.

[2] J. Agler, Hypercontractions and subnormality, J. Operator Theory 13(1985) 203-217.

[3] J. Agler and M. Stankus, m-isometric transformations of Hilbert space I, Integral Equations Operator Theory 21(1995) 383-429.

[4] J. Agler and M. Stankus, m-isometric transformations of Hilbert space III, Integral Equations Operator Theory 24(1996) 379-421.

[5] A. Athavale, On completely hyperexpansive operators, Proc. Math. Soc. 124 (1996) 3745-3752.

[6] F. Bayart, m-isometries on Banach spaces, Math. Nachr. 284(2011) 2141-2147.

[7] C. Berg, J.P.R. Christensen and P. Ressel, Harmonic analysis on semigroup, Springer Verlag, Berlin 1984.

[8] F. Botelho, On the existence of n-isometries on l_p spaces, Acta Sci. Math. (Szeged) 76(2010) 183-192.

[9] L. de Branges and J. Rovnyak, Canonical models in quantum scattering theory, in Perturbation theory and its applications in quantum mechanics, Wiley, New York, 1966, pp 347-392.

[10] C. Foias, A remark on the universal model for contractions of G. C. Rota, Comm. Acad. R. P. Române, 13(1963) 349-352.

[11] H.W. Gould, www.math.wvu.edu/~gould/Vol.4.PDF, Combinatorial identities, 2010.

[12] Caixing Gu, On (m,p)-expansive and (m,p)-contractive operators on Hilbert and Banach spaces, to appear in J. Math. Anal. Appl..

[13] Caixing Gu, The (m,q)-isometric weighted shifts on l_p spaces, to appear in Integral Equations Operator Theory.

[14] P. Hoffman, M. Mackey, and M. Ó Searcóid, On the second parameter of an (m,p)-isometry, Integral Equations Operator Theory 71 (2011) 389-405.

[15] C. S. Kubrusly, An introduction to models and decompositions in operator theory, Birkäuser Boston, 1997.

[16] B. Sz-Nagy and C. Foias, Sur les contractions de l'espace de Hilbert, VIII, Acta Sci. Math (Szeged), 25(1964), 38-71.

[17] G.C. Rota, On models for linear operators, Comm. Pure Appl. Math., 13(1960) 469-472.